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DARBOUX FRAMES OF BERTRAND CURVES IN THE GALILEAN AND PSEUDO-GALILEAN SPACES

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Abstract

This paper aims to define and provide relationships between pairs of Bertrand curves on ruled surfaces of type I in the Galilean and pseudo-Galilean spaces. Moreover, some important conditions between the curvatures of these curves are obtained. Finally, two examples are given to confirm our results.

1. Introduction

Discovering Galilean space-time is probably one of the major achievements of non-relativistic physics. Nowadays, Galilean space is becoming increasingly popular as evidenced from the connection of the fundamental concepts such as velocity, momentum, kinetic energy, etc. and principles as indicated in [1]. In recent years, researchers have begun to investigate curves and surfaces in the Galilean space and thereafter pseudo-Galilean space. Differential geometry of the Galilean space G^3 and especially the geometry of ruled surfaces in this space has been largely developed in

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[2]. Some more results about ruled surfaces in G^3 have been given in [3]. Pseudo-Galilean space G_1^3 has been explained in details in [4-6]. Bertrand curves were discovered by Bertrand in 1850 and since then there were many attempts to generalize either the original defining property or that of the linear relation between curvature and torsion or one looked at generalizations of the embedding space, see [6-14]. In [11], Gluck has investigated the Bertrand curves in n-dimensional Euclidean space E^n. The corresponding characterizations of the Bertrand curves in n-dimensional Lorentzian space L^n have been given by Tosun and Özgür [9]. In [6, 7], the authors studied pairs of non-isotropic Bertrand curves in pseudo-Euclidean 3-space M_1^3 (Minkowski-3-space or, abbreviated, pe-space). Izumiya and Takeuchi [12, 13] considered generic properties of Bertrand curves and their curvature relationship whereas Schief [8] gave a study of the integrability of Bertrand curves. Also, the authors in [10] gave some important characterizations for these types of curves. Furthermore, a number of differential geometers studied this subject for such curves lying on a surface in the Minkowski 3-space M_1^3, see, e.g., [15-17]. This study puts some light onto the pair of Bertrand curves which lie on ruled surfaces of special type (type I) in the Galilean and pseudo-Galilean spaces. We will deal with Darboux frame of non-isotropic curves which depends on the pseudo-Galilean geometry of the supporting ruled surface of these curves and allows to formulate geometric properties of Bertrand pair as the relations between their normals, respectively, geodesic curvatures and torsions. Although, there are a lot of works on Bertrand curves but there are rather a few works on them in the Galilean and pseudo-Galilean spaces. However, to the best of author's knowledge, Bertrand curves have not presented Galilean and pseudo-Galilean spaces in depth. Thus, the study is proposed to serve such a need.

2. Preliminaries

We now review some basic concepts on classical differential geometry of space curves in the Galilean and pseudo-Galilean spaces. The Galilean space is a three-dimensional complex projective space P_3 in which the absolute figure of the Galilean space is the ordered triple $\{w, f, I\}$, where w is the ideal absolute plane in the real three-dimensional projective space $P_3(R)$, f the absolute line in w and I the fixed hyperbolic involution of points of f. Homogeneous coordinates in G^3 are introduced in such a way that the absolute plane w is given by $x_0 = 0$, the absolute line f by $x_0 = x_1 = 0$ and the hyperbolic involution by $(0:0:x_2:x_3) \to (0:0:x_3:x_2)$. The last condition is equivalent to the requirement that the conic $x_2^2 - x_3^2 = 0$ is the absolute conic. Metric relations are introduced with respect to the absolute figure. Throughout this work, we denote the inner and cross products of two vectors P, Q in the sense of Galilean by the notation $\langle P, Q\rangle_{G^3}$ and $(P \times Q)_{G^3}$. In affine coordinates, the Galilean inner product between two vectors $P = (p_1, p_2, p_3)$ and $Q = (q_1, q_2, q_3)$ is defined by [12]:

$$\langle P, Q\rangle_{G^3} = \begin{cases} p_1q_1 & \text{if } p_1 \neq 0 \vee q_1 \neq 0, \\ p_2q_2 + p_3q_3 & \text{if } p_1 = 0 \wedge q_1 = 0 \end{cases} \tag{2.1}$$

whereas the cross product in the sense of Galilean space is given by:

$$(P \times Q)_{G^3} = \begin{cases} \begin{vmatrix} 0 & e_2 & e_3 \\ p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \end{vmatrix}; & \text{if } p_1 \neq 0 \vee q_1 \neq 0, \\ \\ \begin{vmatrix} e_1 & e_2 & e_3 \\ p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \end{vmatrix}; & \text{if } p_1 = 0 \wedge q_1 = 0. \end{cases}$$

A curve $\alpha : I \to G^3$, $I \subset R$ of the class C^r $(r \geq 3)$ in the Galilean space G^3 is defined by the parameterization:

$$\alpha(s) = (s, y(s), z(s)), \tag{2.2}$$

where s is a Galilean invariant parameter (the arc-length on α). The curvature $\kappa_\alpha(s)$ and the torsion $\tau_\alpha(s)$ of α are defined by

$$\kappa_\alpha(s) = \| \alpha''(s) \| = \sqrt{y''(s)^2 + z''(s)^2},$$

$$\tau_\alpha(s) = \frac{Det[\alpha'(s), \alpha''(s), \alpha'''(s)]}{\kappa_\alpha^2(s)}. \tag{2.3}$$

The orthonormal frame in the sense of Galilean geometry is defined by

$$\mathbf{T}(s) = \alpha'(s) = (1, y'(s), z'(s)),$$

$$\mathbf{N}(s) = \frac{\alpha''(s)}{\| \alpha''(s) \|} = \frac{(0, y''(s), z''(s))}{\kappa_\alpha(s)},$$

$$\mathbf{B}(s) = (\mathbf{T}(s) \times \mathbf{N}(s))_{G^3} = \frac{(0, -z''(s), y''(s))}{\kappa_\alpha(s)}. \tag{2.4}$$

The vectors **T**, **N** and **B** in (2.4) are called the *tangent*, *principal normal* and *binormal vectors* of α, respectively. For this frame, the Frenet formulas hold [2],

$$\begin{bmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B} \end{bmatrix}' = \begin{bmatrix} 0 & \kappa_\alpha & 0 \\ 0 & 0 & \tau_\alpha \\ 0 & -\tau_\alpha & 0 \end{bmatrix} \begin{bmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B} \end{bmatrix}. \tag{2.5}$$

The pseudo-Galilean geometry is one of the real Cayley-Klein geometries equipped with the projective metric of signature $(0, 0, +, -)$ as in [4]. The absolute figure of the pseudo-Galilean geometry consists of an ordered triple (Π, L, I), where Π is the ideal (absolute) plane, L the real line (absolute line) in Π and I the fixed hyperbolic involution of points of L. A plane is called *pseudo-Euclidean plane* if it contains L, otherwise it is called *isotropic*. The planes $x = const.$ are pseudo-Euclidean planes and so are the planes Π. Other planes are isotropic. A vector $a = (a_1, a_2, a_3)$ is said to be *non-isotropic* if $a_1 \neq 0$. All unit non-isotropic vectors are of the form $a = (1, a_2, a_3)$. For isotropic vectors $a_1 = 0$ holds. According to the motion group in the pseudo-Galilean space, there are four types of isotropic vectors; spacelike if $a_2^2 - a_3^2 > 0$, timelike if $a_2^2 - a_3^2 < 0$ and two types of lightlike vectors if $a_2 = \pm a_3$. A non-lightlike isotropic vector is a unit vector if $a_2^2 - a_3^2 = \pm 1$. The pseudo-Galilean scalar product of two vectors $p = (p_1, p_2, p_3)$ and $q = (q_1, q_2, q_3)$ is given by

$$\langle p, q\rangle_{G_1^3} = \begin{cases} p_1 q_1 & \text{if } p_1 \neq 0 \vee q_1 \neq 0, \\ p_2 q_2 - p_3 q_3 & \text{if } p_1 = 0 \wedge q_1 = 0. \end{cases}$$

We give a pseudo-Galilean cross product in the following way:

$$(p \times q)_{G_1^3} = \begin{vmatrix} 0 & -j & k \\ p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \end{vmatrix},$$

where $j = (0, 1, 0)$ and $k = (0, 0, 1)$ are unit spacelike and timelike vectors, respectively. If $\omega = (0, p_2, p_3)$ is an isotropic vector, then $p\omega = 0$ (p is orthogonal to ω in the sense of the pseudo-Galilean space) implies $p^2 \neq 0$.

The associated trihedron of the pseudo-Galilean space for a curve $\beta(s)$ is defined by

$$\mathbf{t}(s) = \beta'(s) = (1, y'(s), z'(s)),$$

$$\mathbf{n}(s) = \frac{1}{\kappa_\beta}\beta''(s) = \frac{(0, y''(s), z''(s))}{\kappa_\beta(s)},$$

$$\mathbf{b}(s) = \frac{1}{\kappa_\beta(s)}(0, \varepsilon z''(s), \varepsilon y''(s)), \tag{2.6}$$

where $\varepsilon = \pm 1$, chosen by criterion $\det(\mathbf{t}, \mathbf{n}, \mathbf{b}) = 1$, that means

$$| y''(s)^2 - z''(s)^2 | = \varepsilon(y''(s)^2 - z''(s)^2). \tag{2.7}$$

The curvature and torsion are, respectively, given by

$$\kappa_\beta(s) = \sqrt{| y''(s)^2 - z''(s)^2 |}, \tag{2.8}$$

$$\tau_\beta(s) = \frac{y''(s)z'''(s) - y'''(s)z''(s)}{\kappa_\beta(s)^2}. \tag{2.9}$$

The curve β is timelike (resp. spacelike) if $\mathbf{n}(s)$ is a spacelike (resp. timelike) vector. The principal normal vector or simply normal is spacelike if $\varepsilon = 1$ and timelike if $\varepsilon = -1$. The following Frenet's formula holds:

$$\begin{bmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{bmatrix}' = \begin{bmatrix} 0 & \kappa_\beta & 0 \\ 0 & 0 & \tau_\beta \\ 0 & \tau_\beta & 0 \end{bmatrix} \begin{bmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{bmatrix}. \tag{2.10}$$

A general equation of a ruled C^r-surface; $r \geq 1$ in $G^3(G_1^3)$ is given by its parameterization [18, 19],

$$\Psi(s, v) = \beta(s) + v\mathbf{A}(s); \quad v \in \mathcal{R}. \tag{2.11}$$

In this parameterization, the curve β is the directrix parameterized by the pseudo-Galilean arc-length and $\mathbf{A}$ is a unit generator vector field. The striction curve is defined as the curve of central points, i.e., of points that are the meeting points on a given generator of a common perpendicular on that generator and its near-by generator. We say that the ruled surface given by (2.11) is *regular* if

$$\Psi_s = \frac{\partial \Psi}{\partial s} \neq 0, \quad \Psi_v = \frac{\partial \Psi}{\partial v} \neq 0, \quad \Psi_s \times \Psi_v \neq 0.$$

In what follows, according to the position of the striction curve with respect to the absolute figure of $G^3(G_1^3)$, we investigate Bertrand curves of ruled surfaces of type I in $G^3(G_1^3)$.

Definition 2.1. Let $\Psi(s, v)$ and $\overline{\Psi}(\bar{s}, \bar{v})$ be two ruled surfaces in three-dimensional Galilean (pseudo-Galilean) space $G^3(G_1^3)$ and consider the arc-length parameter curves $\alpha(s)$ and $\overline{\alpha}(\bar{s})$ lying fully on Ψ and $\overline{\Psi}$, respectively. Denote the Darboux frames of α and $\overline{\alpha}$ by $\{\mathbf{T}, \mathbf{U}, \mathbf{V}\}$ and $\{\overline{\mathbf{T}}, \overline{\mathbf{U}}, \overline{\mathbf{V}}\}$, respectively. If there exists a corresponding relationship between the curves α and $\overline{\alpha}$ such that, at the corresponding points of these curves, the Darboux frame element $\mathbf{V}$ of α coincides with the Darboux frame element $\overline{\mathbf{V}}$

of $\overline{\alpha}$, then α is called a *Bertrand curve* and $\overline{\alpha}$ is a *Bertrand mate* of α (see Figure 1):

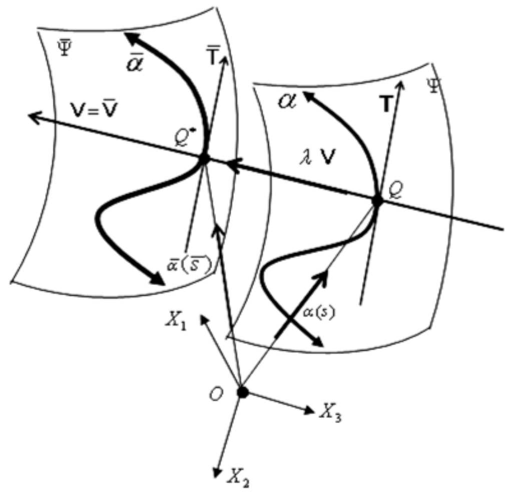

Figure 1. The Bertrand curve α and its Bertrand mate $\overline{\alpha}$.

Definition 2.2. In the differential geometry of surfaces, for a curve $x(s)$ lying on a surface S, the following are well known:

(i) $x(s)$ is a geodesic curve $\Leftrightarrow \kappa_g = 0$,

(ii) $x(s)$ is an asymptotic line $\Leftrightarrow \kappa_n = 0$,

(iii) $x(s)$ is a principal line $\Leftrightarrow \tau_g = 0$ [20].

3. Bertrand Curves of Ruled Surfaces of Type I in G^3

In this section, in G^3, we study Bertrand curves of ruled surfaces of type I as striction curves which do not lie in a pseudo-Euclidean plane and generators $\mathbf{A}(s)$ are non-isotropic. From equation (2.11), we can write

$$\Psi(s, v) = (s, y(s), z(s)) + v(1, \mathbf{A}_2(s), \mathbf{A}_3(s)), \quad v \in \mathcal{R}. \tag{3.1}$$

The associated moving orthonormal trihedron of the ruled surface (3.1) is defined by

$$\mathbf{T}(s) = \mathbf{A}(s),$$

$$\mathbf{N}(s) = \frac{1}{\kappa_\alpha(s)}\mathbf{A}'(s),$$

$$\mathbf{B}(s) = \frac{1}{\kappa_\alpha(s)}(0, -\mathbf{A}'_3(s), \mathbf{A}'_2(s)), \tag{3.2}$$

where κ is the curvature of the ruled surface Ψ and given by

$$\kappa_\alpha(s) = \sqrt{\mathbf{A}'^2_2 + \mathbf{A}'^2_3}. \tag{3.3}$$

The function

$$\tau_\alpha(s) = \frac{\det(\mathbf{A}(s), \mathbf{A}'(s), \mathbf{A}''(s))}{\kappa_\alpha^2} \tag{3.4}$$

is the torsion of Ψ. The surface frame $\{\mathbf{T}, \mathbf{U}, \mathbf{V}\}$ is defined by

$$\mathbf{T} = \mathbf{A}(s), \quad \mathbf{U} = \frac{\Psi_s \times \Psi_v}{|\Psi_s \times \Psi_v|}, \quad \mathbf{V} = \mathbf{U} \times \mathbf{T}; \quad \langle \mathbf{T}, \mathbf{V}\rangle = \langle \mathbf{U}, \mathbf{V}\rangle = 0, \tag{3.5}$$

where $\mathbf{U}$ is the isotropic normal vector of Ψ. Let ϕ be the angle between the isotropic vectors $\mathbf{U}$ and $\mathbf{N}$. Then we may express the results in matrix form as follows:

$$\begin{bmatrix} \mathbf{T} \\ \mathbf{V} \\ \mathbf{U} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B} \end{bmatrix}. \tag{3.6}$$

By straightforward calculations, we have the Darboux frame equations as follows:

$$\frac{d}{ds}\begin{bmatrix} \mathbf{T} \\ \mathbf{V} \\ \mathbf{U} \end{bmatrix} = \begin{bmatrix} 0 & (\kappa_n)_\alpha & (\kappa_g)_\alpha \\ 0 & 0 & (\tau_g)_\alpha \\ 0 & -(\tau_g)_\alpha & 0 \end{bmatrix} \begin{bmatrix} \mathbf{T} \\ \mathbf{V} \\ \mathbf{U} \end{bmatrix}, \tag{3.7}$$

where $(\kappa_n)_\alpha$, $(\kappa_g)_\alpha$ and $(\tau_g)_\alpha$ are the normal curvature, geodesic curvature and the relative torsion, respectively, given by

$$(\kappa_n)_\alpha = \kappa_\alpha \cos\phi, \quad (\kappa_g)_\alpha = -\kappa_\alpha \sin\phi, \quad (\tau_g)_\alpha = \left(\tau_\alpha + \frac{d\phi}{ds}\right). \tag{3.8}$$

Now, by considering the Darboux frame, we give some characterizations of Bertrand curves in G^3 through the following theorems.

Theorem 3.1. *Let* Ψ *and* $\overline{\Psi}$ *be two ruled surfaces of type* I *with* α *and* $\overline{\alpha}$ *their striction curves*, *respectively*, *in Galilean space* G^3 *given by*

$$\Psi(s, v) = \alpha(s) + v\mathbf{A}(s), \quad \overline{\Psi}(\bar{s}, \bar{v}) = \overline{\alpha}(\bar{s}) + \bar{v}\mathbf{B}(\bar{s}).$$

Then $\overline{\alpha}$ *is a Bertrand mate of* α *if and only if the curve* α *is a geodesic curve*.

Proof. Firstly, if α and $\overline{\alpha}$ are the Bertrand and Bertrand mate curves of Ψ and $\overline{\Psi}$, respectively, then by the definition, we can write

$$\overline{\alpha}(\bar{s}) = \alpha(s) + \lambda(s)\mathbf{V}. \tag{3.9}$$

Differentiating (3.9) with respect to s and using equations (3.7), one can obtain

$$\overline{\mathbf{T}}\frac{d\bar{s}}{ds} = \mathbf{T} + \dot{\lambda}(s)\mathbf{V}(s) + \lambda(\tau_g)_\alpha \mathbf{U}. \tag{3.10}$$

Since the direction of $\mathbf{V}$ coincides with the direction of $\overline{\mathbf{V}}$, we get $\dot{\lambda}(s) = 0$. This means that λ is a nonzero constant. Thus, (3.10) becomes

$$\overline{\mathbf{T}}\frac{d\bar{s}}{ds} = \mathbf{T} + \lambda(\tau_g)_\alpha \mathbf{U}. \tag{3.11}$$

Furthermore, we have

$$\overline{\mathbf{T}} = \cos\theta\mathbf{T} + \sin\theta\mathbf{U}, \tag{3.12}$$

where θ is the angle between the tangent vectors $\mathbf{T}$ and $\overline{\mathbf{T}}$ at the corresponding points of α and $\overline{\alpha}$. By differentiating (3.12) with respect to s, we get

$$((\bar{\kappa}_n)_\alpha \overline{\mathbf{V}} + (\bar{\kappa}_g)_\alpha \overline{\mathbf{U}}) \frac{d\bar{s}}{ds} = -\dot{\theta} \sin\theta \mathbf{T} + ((\kappa_n)_\alpha \cos\theta - (\tau_g)_\alpha \sin\theta)\mathbf{V}$$

$$+ ((\kappa_g)_\alpha \cos\theta + \dot{\theta}\cos\theta)\mathbf{U}. \tag{3.13}$$

Also, we can write

$$\overline{\mathbf{U}} = -\sin\theta \mathbf{T} + \cos\theta \mathbf{U}. \tag{3.14}$$

From (3.13) and (3.14), we get

$$((\bar{\kappa}_n)_\alpha \overline{\mathbf{V}} + (\bar{\kappa}_g)_\alpha (-\sin\theta \mathbf{T} + \cos\theta \mathbf{U})) \frac{d\bar{s}}{ds}$$

$$= -\dot{\theta} \sin\theta \mathbf{T} + ((\kappa_n)_\alpha \cos\theta - (\tau_g)_\alpha \sin\theta)\mathbf{V}$$

$$+ ((\kappa_g)_\alpha \cos\theta + \dot{\theta}\cos\theta)\mathbf{U}. \tag{3.15}$$

Equating the coefficients of $\mathbf{T}$, $\mathbf{U}$ on both sides of (3.15), we obtain

$$(\kappa_g)_\alpha = 0. \tag{3.16}$$

Then the striction curve on the ruled surface Ψ is a geodesic curve.

Secondly, if the striction curve α is a geodesic curve, then from (3.11), we get

$$\left(\frac{d\bar{s}}{ds}\right)^2 = 1 + \lambda^2 (\tau_g)_\alpha^2, \tag{3.17}$$

it follows that

$$\lambda(\dot{\tau}_g)_\alpha = \left(\frac{d\bar{s}}{ds}\right)^3 (\bar{\kappa}_g)_\alpha. \tag{3.18}$$

We proceed as above, differentiating (3.11) gives

$$((\bar{\kappa}_n)_\alpha \overline{\mathbf{V}} + (\bar{\kappa}_g)_\alpha \overline{\mathbf{U}}) \left(\frac{d\bar{s}}{ds}\right)^2 + \overline{\mathbf{T}} \frac{d^2\bar{s}}{ds^2} = ((\kappa_n)_\alpha - \lambda(\tau_g)_\alpha^2)\mathbf{V} + (\lambda(\dot{\tau}_g)_\alpha)\mathbf{U}.$$

(3.19)

Taking the cross product of (3.11) with (3.19) to get

$$((\overline{\kappa}_n)_\alpha \overline{\mathbf{U}} - (\overline{\kappa}_g)_\alpha \overline{\mathbf{V}})\left(\frac{d\overline{s}}{ds}\right)^3$$

$$= -\lambda(\tau_g)_\alpha((\kappa_n)_\alpha - \lambda(\tau_g)_\alpha^2)\mathbf{T} - (\lambda(\dot{\tau}_g)_\alpha)\mathbf{V} + ((\kappa_n)_\alpha - \lambda(\tau_g)_\alpha^2)\mathbf{U}. \quad (3.20)$$

Substituting (3.18) into (3.20), we get

$$((\overline{\kappa}_g)_\alpha \overline{\mathbf{U}} + (\overline{\kappa}_n)_\alpha \overline{\mathbf{V}})\left(\frac{d\overline{s}}{ds}\right)^3$$

$$= -\lambda(\tau_g)_\alpha((\kappa_n)_\alpha - \lambda(\tau_g)_\alpha^2)\mathbf{T} - \left((\overline{\kappa}_g)_\alpha\left(\frac{d\overline{s}}{ds}\right)^3\right)\mathbf{V} + ((\kappa_n)_\alpha - \lambda(\tau_g)_\alpha^2)\mathbf{U}.$$

$$(3.21)$$

Also, the cross product of (3.11) with (3.21) yields

$$(-(\overline{\kappa}_n)_\alpha \overline{\mathbf{V}} - (\overline{\kappa}_g)_\alpha \overline{\mathbf{U}})\left(\frac{d\overline{s}}{ds}\right)^4$$

$$= \lambda(\tau_g)_\alpha\left((\overline{\kappa}_g)_\alpha\left(\frac{d\overline{s}}{ds}\right)^3\right)\mathbf{T}$$

$$-(((\kappa_n)_\alpha - \lambda(\tau_g)_\alpha^2) + \lambda^2(\tau_g)_\alpha^2((\kappa_n)_\alpha - \lambda(\tau_g)_\alpha^2))\mathbf{V} \quad (3.22)$$

$$-\left((\overline{\kappa}_g)_\alpha\left(\frac{d\overline{s}}{ds}\right)^3\right)\mathbf{U}.$$

Now, the norm of (3.21) is given by

$$(\overline{\kappa}_n)_\alpha\left(\frac{d\overline{s}}{ds}\right)^3 = \left(\frac{d\overline{s}}{ds}\right)((\kappa_n)_\alpha - \lambda(\tau_g)_\alpha^2). \quad (3.23)$$

Multiplying (3.20) with $(\overline{\kappa}_n)_\alpha \dfrac{d\overline{s}}{ds}$ and (3.21) with $\overline{\kappa}_g$ and adding them, we get

$$((\bar{\kappa}_n)_\alpha^2 + (\bar{\kappa}_g)_\alpha^2)\left(\frac{d\bar{s}}{ds}\right)^4 \overline{\mathbf{U}}$$

$$= -\lambda(\tau_g)_\alpha\left[((\kappa_n)_\alpha - \lambda(\tau_g)_\alpha^2)(\bar{\kappa}_n)_\alpha \frac{d\bar{s}}{ds} + (\bar{\kappa}_g)_\alpha^2\left(\frac{d\bar{s}}{ds}\right)^3\right]\mathbf{T}$$

$$+\left[-(\bar{\kappa}_n)_\alpha \frac{d\bar{s}}{ds}\left((\bar{\kappa}_g)_\alpha\left(\frac{d\bar{s}}{ds}\right)^3\right) + (\bar{\kappa}_g)_\alpha((\kappa_n)_\alpha - \lambda(\tau_g)_\alpha^2)\left(\frac{d\bar{s}}{ds}\right)^2\right]\mathbf{V}$$

$$+\left[(\bar{\kappa}_n)_\alpha \frac{d\bar{s}}{ds}((\kappa_n)_\alpha - \lambda(\tau_g)_\alpha^2) + (\bar{\kappa}_g)_\alpha^2\left(\frac{d\bar{s}}{ds}\right)^3\right]\mathbf{U}. \tag{3.24}$$

Inserting (3.22) into (3.24), we obtain

$$((\bar{\kappa}_n)_\alpha^2 + (\bar{\kappa}_g)_\alpha^2)\left(\frac{d\bar{s}}{ds}\right)^4 \overline{\mathbf{U}} = -\lambda(\tau_g)_\alpha\left[(\bar{\kappa}_n)_\alpha^3\left(\frac{d\bar{s}}{ds}\right)^3 + (\bar{\kappa}_g)_\alpha^2\left(\frac{d\bar{s}}{ds}\right)^3\right]\mathbf{T}$$

$$+\left[(\bar{\kappa}_n)_\alpha^3\left(\frac{d\bar{s}}{ds}\right)^3 + (\bar{\kappa}_g)_\alpha^2\left(\frac{d\bar{s}}{ds}\right)^3\right]\mathbf{U}. \tag{3.25}$$

In the light of the above, equations (3.11) and (3.25) show that the vectors $\overline{\mathbf{T}}$ and $\overline{\mathbf{U}}$ lie in the plane which contains $\mathbf{T}$ and $\mathbf{U}$. Thus, the vectors $\mathbf{V}$ and $\overline{\mathbf{V}}$ are coincident at the corresponding points. Hence, the proof is completed. □

Theorem 3.2. *Let* α *and* $\bar{\alpha}$ *be Bertrand and its Bertrand mate curves of two ruled surfaces of type* I. *Then the product of relative torsions* $(\tau_g)_\alpha$ *and* $(\bar{\tau}_g)_\alpha$ *is constant.*

Proof. As $\bar{\alpha}$ is the Bertrand mate of α, one can write

$$\alpha(s) = \bar{\alpha}(\bar{s}) - \lambda\overline{\mathbf{V}}; \quad \mathbf{V} = \overline{\mathbf{V}}. \tag{3.26}$$

By differentiating (3.26) with respect to s, we get

$$\mathbf{T} = \frac{d\bar{s}}{ds}\overline{\mathbf{T}} - \lambda(\bar{\tau}_g)_\alpha \frac{d\bar{s}}{ds}\overline{\mathbf{U}}. \tag{3.27}$$

From (3.12) and (3.14), we have

$$\mathbf{T} = \cos\theta\overline{\mathbf{T}} - \sin\theta\overline{\mathbf{U}}. \tag{3.28}$$

So, equations (3.27) and (3.28) together yield

$$\cos\theta = \frac{d\bar{s}}{ds},$$

$$\sin\theta = \lambda(\bar{\tau}_g)_\alpha \frac{d\bar{s}}{ds}. \tag{3.29}$$

Furthermore, from (3.11) and (3.12), the following equality holds:

$$\frac{d\bar{s}}{ds} = \frac{1}{\cos\theta} = \frac{\lambda(\tau_g)_\alpha}{\sin\theta}, \tag{3.30}$$

it follows that

$$\tan\theta = \lambda(\tau_g)_\alpha. \tag{3.31}$$

Thus, equations (3.29) and (3.31) lead to the desired result. □

Proposition 3.1. *Let* $\{\alpha, \overline{\alpha}\}$ *be a Bertrand pair of ruled surfaces of type* I *in* G^3. *Then the curvatures of* α *and* $\overline{\alpha}$ *satisfy the following relations*:

(i) $(\cos\theta)\,(\kappa_n)_\alpha - (\sin\theta)\,(\tau_g)_\alpha = 0,$

(ii) $(\bar{\tau}_g)_\alpha = \left(\frac{ds}{d\bar{s}}\right)^2 ((\kappa_n)_\alpha \sin\theta + (\tau_g)_\alpha \cos\theta)(\lambda(\tau_g)_\alpha \sin\theta + \cos\theta).$

Proof. (i) As $\mathbf{V}$ and $\overline{\mathbf{V}}$ are coincidences, the differentiation of (3.5) with respect to $\bar{s}$ gives

$$\langle \mathbf{U}, (\bar{\tau}_g)_\alpha \overline{\mathbf{U}} \rangle + \left\langle -(\tau_g)_\alpha \mathbf{V} \frac{ds}{d\bar{s}}, \overline{\mathbf{V}} \right\rangle = 0 \tag{3.32}$$

and

$$\langle \mathbf{T}, (\bar{\tau}_g)_\alpha \overline{\mathbf{U}} \rangle + \left\langle ((\kappa_n)_\alpha \mathbf{V} + (\kappa_g)_\alpha \mathbf{U}) \frac{ds}{d\bar{s}}, \overline{\mathbf{V}} \right\rangle = 0. \tag{3.33}$$

Inserting (3.14) into (3.32) and (3.33) and using Theorem 3.1, we get

$$(\bar{\tau}_g)_\alpha = (\tau_g)_\alpha \sec\theta \frac{ds}{d\bar{s}}, \quad (\kappa_n)_\alpha = (\bar{\tau}_g)_\alpha \sin\theta \frac{d\bar{s}}{ds}. \tag{3.34}$$

Equation (3.34) leads to

$$(\cos\theta)(\kappa_n)_\alpha - (\sin\theta)(\tau_g)_\alpha = 0.$$

It is known that, from classical differential geometry, the geodesic curvature and geodesic torsion of a curve on a surface are given by [20],

$$(\bar{\kappa}_g)_\alpha = \left\langle \frac{d\bar{\alpha}}{d\bar{s}}, \frac{d^2\bar{\alpha}}{d\bar{s}^2} \times \bar{\mathbf{U}} \right\rangle, \quad (\bar{\tau}_g)_\alpha = \left\langle \frac{d\bar{\alpha}}{d\bar{s}}, \bar{\mathbf{U}} \times \frac{d\bar{\mathbf{U}}}{d\bar{s}} \right\rangle, \tag{3.35}$$

where $\bar{\mathbf{U}}$ being the surface normal, then from (3.9) and (3.14), we get

$$\frac{d\bar{\alpha}}{d\bar{s}} = \mathbf{T}\frac{ds}{d\bar{s}} + \lambda(\tau_g)_\alpha \frac{ds}{d\bar{s}}\mathbf{U}, \tag{3.36}$$

$$\begin{aligned}\frac{d\bar{\mathbf{U}}}{d\bar{s}} &= -\left(\frac{d\theta}{d\bar{s}}\cos\theta\right)\mathbf{T} - \left(\frac{ds}{d\bar{s}}\right)((\kappa_n)_\alpha \sin\theta + (\tau_g)_\alpha \cos\theta)\mathbf{V} \\ &\quad - \sin\theta\left(\frac{d\theta}{d\bar{s}} + (\kappa_g)_\alpha \frac{ds}{d\bar{s}}\right)\mathbf{U}.\end{aligned} \tag{3.37}$$

Using the cross product of (3.14) with (3.37), we get

$$\begin{aligned}\bar{\mathbf{U}} \times \frac{d\bar{\mathbf{U}}}{d\bar{s}} &= \cos\theta\left(\frac{ds}{d\bar{s}}\right)((\kappa_n)_\alpha \sin\theta + (\tau_g)_\alpha \cos\theta)\mathbf{T} \\ &\quad - \left(\left(\frac{d\theta}{d\bar{s}}\cos^2\theta\right) + \sin^2\theta\left(\frac{d\theta}{d\bar{s}} - (\kappa_g)_\alpha \frac{ds}{d\bar{s}}\right)\right)\mathbf{V} \\ &\quad + b\sin\theta\left(\frac{ds}{d\bar{s}}\right)((\kappa_n)_\alpha \sin\theta + (\tau_g)_\alpha \cos\theta)\mathbf{U}.\end{aligned} \tag{3.38}$$

Thus, the scalar product of (3.36) with (3.38) satisfies (ii). □

Under the previous proposition, we can discuss the following cases:

Case 3.1. If $\alpha(s)$ is an asymptotic line, then the linear relation

$$(\lambda(\tau_g)_\alpha)(\overline{\tau}_g)_\alpha + ((\sqrt{1 + \lambda^2(\tau_g)_\alpha^2})\cos\theta)(\overline{\kappa}_n)_\alpha = 0 \tag{3.39}$$

is satisfied.

Case 3.2. If $\alpha(s)$ is a principal line, then we get

$$(\overline{\tau}_g)_\alpha - ((\sqrt{1 + \lambda^2(\tau_g)_\alpha^2})\sin\theta)(\overline{\kappa}_n)_\alpha = 0. \tag{3.40}$$

Notation 3.1. The above two cases together with equations (3.17) and (3.29) give the following important relations:

$$(\overline{\tau}_g)_\alpha = \frac{(\tau_g)_\alpha}{(1 - \lambda^2(\tau_g)_\alpha^2)}, \quad (\overline{\kappa}_n)_\alpha = \frac{(\kappa_n)_\alpha}{(1 + \lambda^2(\tau_g)_\alpha^2)}. \tag{3.41}$$

It follows that the curve $\overline{\alpha}$ is a helix in G^3 if α is too.

4. Bertrand Curves of Ruled Surface of Type I in G_1^3

In this section, we study Bertrand curves on ruled C^r-surfaces; $r \gg 1$ of type I (its curve $\beta(s)$ does not lie in a pseudo-Euclidean plane and the generators $\mathbf{A}(s)$ are non-isotropic) in G_1^3 given by the parameterization (3.1). In the sense of G_1^3, the associated moving orthonormal trihedron of this ruled surface is defined by

$$\mathbf{t}(s) = \mathbf{A}(s),$$

$$\mathbf{n}(s) = \frac{1}{\kappa_\beta}\mathbf{A}'(s),$$

$$\mathbf{b}(s) = \frac{1}{\kappa_\beta}(0, \mathbf{A}_3'(s), \mathbf{A}_2'(s)), \tag{4.1}$$

where κ_β is the curvature of Φ given by

$$\kappa_\beta = \sqrt{|\mathbf{A}_2'^2 - \mathbf{A}_3'^2|}, \tag{4.2}$$

and the function

$$\tau_\beta = \frac{\det(\mathbf{A}(s),\ \mathbf{A}'(s),\ \mathbf{A}''(s))}{\kappa_\beta^2} \tag{4.3}$$

is the torsion of Φ. Let φ be the hyperbolic angle between the isotropic timelike vectors **U** and **n**. Then we may express the results in matrix form as follows:

$$\begin{bmatrix} \mathbf{t} \\ \mathbf{U} \\ \mathbf{V} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cosh\varphi & \sinh\varphi \\ 0 & \sinh\varphi & \cosh\varphi \end{bmatrix} \begin{bmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{bmatrix}. \tag{4.4}$$

By a straightforward computation, we have the Darboux frame equations as follows:

$$\frac{d}{ds}\begin{bmatrix} \mathbf{t} \\ \mathbf{U} \\ \mathbf{V} \end{bmatrix} = \begin{bmatrix} 0 & (\kappa_n)_\beta & (\kappa_g)_\beta \\ 0 & 0 & (\tau_g)_\beta \\ 0 & (\tau_g)_\beta & 0 \end{bmatrix} \begin{bmatrix} \mathbf{t} \\ \mathbf{U} \\ \mathbf{V} \end{bmatrix}, \tag{4.5}$$

where $(\kappa_n)_\beta$, $(\kappa_g)_\beta$ and $(\tau_g)_\beta$ are the normal curvature, the geodesic curvature and the relative torsion, respectively, given by

$$(\kappa_n)_\beta = \kappa_\beta \cosh\varphi, \quad (\kappa_g)_\beta = -\kappa_\beta \sinh\varphi, \quad (\tau_g)_\beta = \left(\tau_\beta + \frac{d\varphi}{ds}\right). \tag{4.6}$$

Now, by considering the Darboux frame, we define Bertrand curves and give the characterizations of these curves in pseudo-Galilean space G_1^3 through the following theorems.

Theorem 4.1. *Let* Φ *and* $\overline{\Phi}$ *be two ruled surfaces of type* I *and* β, $\overline{\beta}$ *their striction curves in* G_1^3, *respectively, given by*

$$\Phi(s, v) = \beta(s) + v\mathbf{A}(s), \quad \overline{\Phi}(\bar{s}, \bar{v}) = \overline{\beta}(\bar{s}) + \bar{v}\mathbf{B}(\bar{s}).$$

Then $\overline{\beta}$ *is a Bertrand mate of* β *if and only if the curve* β *is an asymptotic line*.

Proof. The proof is similar to that considered in Theorem 3.1 with $(\kappa_n)_\beta = 0.$ □

Theorem 4.2. *Let* β *and* $\overline{\beta}$ *be Bertrand and its Bertrand mate curves of two ruled surfaces of type* I. *Then the product of their relative torsions* $(\tau_g)_\beta$ *and* $(\bar{\tau}_g)_\beta$ *is constant*.

Proof. Similar to the proof of Theorem 3.2. □

Proposition 4.1. *Let* $\{\beta, \overline{\beta}\}$ *be a Bertrand pair of ruled surfaces of type* I *in* G_1^3. *Then the curvatures of* β *and* $\overline{\beta}$ *satisfy*, *respectively*,

(i) $(\cosh\Theta)(\kappa_g)_\beta + (\sinh\Theta)(\tau_g)_\beta = 0,$

where Θ *is the angle between the tangent vectors* $\mathbf{t}$ *and* $\bar{\mathbf{t}}$ *at the corresponding points of* β *and* $\overline{\beta}$,

(ii) $(\bar{\kappa}_g)_\beta = \left(\dfrac{ds}{d\bar{s}}\right)^3((\kappa_g)_\beta + \lambda(\tau_g)_\beta^2)(\cosh\Theta - \lambda(\tau_g)_\beta \sinh\Theta),$

(iii) $(\bar{\tau}_g)_\beta = \left(\dfrac{ds}{d\bar{s}}\right)^2((\kappa_g)_\beta \sinh\Theta + (\tau_g)_\beta \cosh\Theta)(\lambda(\tau_g)_\beta \sinh\Theta - \cosh\Theta).$

Proof. As $\mathbf{V}$ and $\overline{\mathbf{V}}$ are coincidence, the differentiation of (3.5) with respect to $\bar{s}$ gives

$$\langle \mathbf{U}, (\bar{\tau}_g)_{\overline{\beta}}\overline{\mathbf{U}}\rangle + \left\langle (\tau_g)_\beta \mathbf{V}\frac{ds}{d\bar{s}}, \overline{\mathbf{V}}\right\rangle = 0 \tag{4.7}$$

and

$$\langle \mathbf{t}, (\bar{\tau}_g)_{\overline{\beta}}\overline{\mathbf{U}}\rangle + \left\langle ((\kappa_g)_\beta \mathbf{V} + (\kappa_n)_\beta \mathbf{U})\frac{ds}{d\bar{s}}, \overline{\mathbf{V}}\right\rangle = 0. \tag{4.8}$$

Also, we can write

$$\overline{\mathbf{U}} = \sinh\Theta\mathbf{t} + \cosh\Theta\mathbf{U}. \tag{4.9}$$

Inserting (4.9) into (4.7) and (4.8) and using Theorem 4.1, we get, respectively,

$$(\bar{\tau}_g)_{\bar{\beta}} = (\tau_g)_\beta \operatorname{sech}\Theta \frac{ds}{d\bar{s}}, \quad (\kappa_g)_\beta = -(\bar{\tau}_g)_{\bar{\beta}} \sinh\Theta \frac{d\bar{s}}{ds}. \tag{4.10}$$

Equation (4.10) leads to

$$(\cosh\Theta)(\kappa_g)_\beta + (\sinh\Theta)(\tau_g)_\beta = 0.$$

Now, in the light of (3.9); $\overline{\beta}(\bar{s}) = \beta(s) + \lambda(s)\mathbf{V}$, one can calculate the following:

$$\frac{d\overline{\beta}}{d\bar{s}} = \mathbf{t}\frac{ds}{d\bar{s}} + \lambda(\tau_g)_\beta \frac{ds}{d\bar{s}}\mathbf{U}, \tag{4.11}$$

$$\frac{d^2\overline{\beta}}{d\bar{s}^2} = \frac{d^2 s}{d\bar{s}^2}\mathbf{T} + \left(\left(\frac{ds}{d\bar{s}}\right)^2((\kappa_n)_\beta + \lambda(\dot{\tau}_g)_\beta) + \lambda(\tau_g)_\beta \frac{d^2 s}{d\bar{s}^2}\right)\mathbf{U}$$
$$+\left(\left(\frac{ds}{d\bar{s}}\right)^2((\kappa_g)_\beta + \lambda(\tau_g)_\beta^2)\right)\mathbf{V}, \tag{4.12}$$

$$\frac{d\overline{\mathbf{U}}}{d\bar{s}} = \left(\frac{d\Theta}{d\bar{s}}\cosh\Theta\right)\mathbf{T} + \sinh\Theta\left(\frac{d\Theta}{d\bar{s}} + (\kappa_n)_\beta \frac{ds}{d\bar{s}}\right)\mathbf{U}$$
$$+\left(\frac{ds}{d\bar{s}}\right)((\kappa_g)_\beta \sinh\Theta + (\tau_g)_\beta \cosh\Theta)\mathbf{V}. \tag{4.13}$$

Using the cross product of (4.9) with (4.12) and (4.13) to obtain

$$\frac{d^2\overline{\beta}}{d\bar{s}^2} \times \overline{\mathbf{U}} = \cosh\Theta\left(\left(\frac{ds}{d\bar{s}}\right)^2((\kappa_g)_\beta + \lambda(\tau_g)_\beta^2)\right)\mathbf{T}$$
$$+\sinh\Theta\left(\left(\frac{ds}{d\bar{s}}\right)^2((\kappa_g)_\beta + \lambda(\tau_g)_\beta^2)\right)\mathbf{U}$$
$$+\left(-\cosh\Theta\left(\frac{d^2 s}{d\bar{s}^2}\right)\right.$$
$$\left.+\sinh\Theta\left(\left(\frac{ds}{d\bar{s}}\right)^2((\kappa_n)_\beta + \lambda(\dot{\tau}_g)_\beta) + \lambda(\tau_g)_\beta\frac{d^2 s}{d\bar{s}^2}\right)\right)\mathbf{V}, \tag{4.14}$$

$$\overline{\mathbf{U}} \times \frac{d\overline{\mathbf{U}}}{d\bar{s}} = -\cosh\Theta\left(\frac{ds}{d\bar{s}}\right)((\kappa_g)_\beta \sinh\Theta + (\tau_g)_\beta \cosh\Theta)\mathbf{T}$$

$$- \sinh\Theta\left(\frac{ds}{d\bar{s}}\right)((\kappa_g)_\beta \sinh\Theta + (\tau_g)_\beta \cosh\Theta)\mathbf{U}$$

$$+ \left(\left(\frac{d\theta}{d\bar{s}}\cosh^2\Theta\right) - \sinh^2\Theta\left(\frac{d\Theta}{d\bar{s}} + (\kappa_n)_\beta \frac{ds}{d\bar{s}}\right)\right)\mathbf{V}. \tag{4.15}$$

According to (3.35), the scalar product of (4.11) with (4.14) and (4.15) satisfies the other two relations. □

In the light of this proposition, we can discuss the following cases:

Case 4.1. If $\beta(s)$ is a geodesic curve, then the linear relation

$$(\sqrt{1 - \lambda^2(\tau_g)^2_\beta})(\bar{\kappa}_g)_\beta + \lambda(\tau_g)_\beta(\bar{\tau}_g)_\beta = 0 \tag{4.16}$$

is satisfied.

Case 4.2. If $\beta(s)$ is a principal line, then we get the following linear relation:

$$((\sqrt{1 - \lambda^2(\tau_g)^2_\beta})\sinh\Theta)(\bar{\kappa}_g)_\beta + (\bar{\tau}_g)_\beta = 0. \tag{4.17}$$

Notation 4.1. From the previous calculations, we have

$$\left(\frac{d\bar{s}}{ds}\right)^2 = 1 - \lambda^2(\tau_g)^2_\beta, \quad \sinh\theta = \lambda(\bar{\tau}_g)_\beta \frac{d\bar{s}}{ds}. \tag{4.18}$$

Then the above two cases together with equations (4.18) give, respectively,

$$(\bar{\tau}_g)_\beta = -\frac{(\tau_g)_\beta}{(1 - \lambda^2(\tau_g)^2_\beta)}, \quad (\bar{\kappa}_g)_\beta = \frac{(\kappa_g)_\beta}{(1 - \lambda^2(\tau_g)^2_\beta)}, \tag{4.19}$$

it follows that $\bar{\beta}$ is a geodesic helix in G_1^3 if β is too.

Hereafter, we give two examples to investigate our main results.

5. Examples

Example 5.1. Consider the ruled surface of type I in Galilean space G^3 parameterized by

$$\Psi(s, v) = \left(s,\ f\sin\left(\frac{s}{h}\right),\ f\cos\left(\frac{s}{h}\right)\right) + v\left(1,\ \frac{1}{g}\cos\left(\frac{s}{h}\right),\ -\frac{1}{g}\sin\left(\frac{s}{h}\right)\right),$$

where $f, g, h \in \mathcal{R}$, $f \neq 0$, $g \neq 0$, $h \neq 0$ and $|fg - h| > |v|$. Its striction curve $\alpha(s) = \left(s,\ f\sin\left(\frac{s}{h}\right),\ f\cos\left(\frac{s}{h}\right)\right)$ does not lie in a pseudo-Euclidean plane and $A(s) = \left(1,\ \frac{1}{g}\cos\left(\frac{s}{h}\right),\ -\frac{1}{g}\sin\left(\frac{s}{h}\right)\right)$ is a unit generator non-isotropic vector field. From (3.2), (3.3) and (3.4), the Frenet apparatus of $\alpha(s)$ is given by

$$A(s) = \mathbf{T} = \left(1,\ \frac{1}{g}\cos\left(\frac{s}{h}\right),\ -\frac{1}{g}\sin\left(\frac{s}{h}\right)\right),$$

$$\mathbf{N} = \left(0,\ -\sin\left(\frac{s}{h}\right),\ \cos\left(\frac{s}{h}\right)\right),$$

$$\mathbf{B} = \left(0,\ \cos\left(\frac{s}{h}\right),\ -\sin\left(\frac{s}{h}\right)\right),$$

$$\kappa_\alpha = \frac{1}{gh}, \quad \tau_\alpha = -\frac{1}{h}.$$

Moreover, from (3.5), the normal vector of Ψ is expressed as

$$\mathbf{U} = \frac{\left(0,\ (h - fg)\sin\left(\frac{s}{h}\right) - v\cos\left(\frac{s}{h}\right),\ (h - fg)\cos\left(\frac{s}{h}\right) + v\sin\left(\frac{s}{h}\right)\right)}{\sqrt{(h - fg)^2 + v^2}}.$$

If ϕ is the angle between **U** and **N**, then it is given by

$$\langle \mathbf{U}, \mathbf{N} \rangle = |\mathbf{U}||\mathbf{N}|\cos\phi,$$

it follows that

$$\cos\phi = \frac{-(h - fg)}{\sqrt{v^2 + (h - fg)^2}}, \quad \sin\phi = \frac{v}{\sqrt{v^2 + (h - fg)^2}}.$$

Furthermore, by using equation (3.5), we get

$$\mathbf{V} = \frac{\left(0, \left((h - fg)\cos\left(\frac{s}{h}\right) + v\sin\left(\frac{s}{h}\right)\right), -\left((h - fg)\sin\left(\frac{s}{h}\right) - v\cos\left(\frac{s}{h}\right)\right)\right)}{\sqrt{v^2 + (h - fg)^2}}.$$

Here, assuming that $\overline{\alpha}(\overline{s})$ is the Bertrand mate of $\alpha,$ then it can be written as

$$\begin{aligned}\overline{\alpha}(\overline{s}) &= \alpha(s) + \lambda(s)\mathbf{V}\\ &= \left(s,\ f\sin\left(\frac{s}{h}\right),\ f\cos\left(\frac{s}{h}\right)\right)\\ &\quad + \frac{\lambda\left(0, \left((h - fg)\cos\left(\frac{s}{h}\right) + v\sin\left(\frac{s}{h}\right)\right), -\left((h - fg)\sin\left(\frac{s}{h}\right) - v\cos\left(\frac{s}{h}\right)\right)\right)}{\sqrt{v^2 + (h - fg)^2}}.\end{aligned}$$

From equation (3.8), one can obtain

$$(\tau_g)_\alpha = -\frac{1}{h}.$$

Now, as θ is the angle between the tangent vectors $\mathbf{T}$ and $\overline{\mathbf{T}},$ equation (3.29) gives

$$\cos\theta = \frac{ds}{d\overline{s}} = \frac{h}{\sqrt{h^2 + \lambda^2}}, \quad \sin\theta = \lambda(\tau_g)_\alpha \frac{ds}{d\overline{s}} = \frac{-\lambda}{\sqrt{h^2 + \lambda^2}}.$$

Using Proposition 3.1 to obtain

$$(\kappa_n)_\alpha = \frac{\lambda}{h^2}.$$

In addition to the above, the curvatures $(\overline{\tau}_g)_\alpha,$ $(\overline{\kappa}_g)_\alpha$ and $(\overline{\kappa}_n)_\alpha$ of $\overline{\alpha}(\overline{s})$

can be computed from (3.8) as follows:

$$(\bar{\tau}_g)_\alpha = -\frac{h}{(h^2 - \lambda^2)}, \quad (\bar{\kappa}_n)_\alpha = \frac{\lambda}{(h^2 + \lambda^2)}, \quad (\bar{\kappa}_g)_\alpha = (\kappa_g)_\alpha = 0.$$

Then α and $\bar{\alpha}$ are geodesic curves. Finally, the Darboux frame $\{\bar{\mathbf{T}}, \bar{\mathbf{U}}, \bar{\mathbf{V}}\}$ is expressed as:

$$\bar{\mathbf{T}} = \left(\begin{array}{c} \cos\theta, \left(\frac{1}{g}\cos\theta\cos\left(\frac{s}{h}\right) + \sin\theta \frac{(h - fg)\sin\left(\frac{s}{h}\right) - v\cos\left(\frac{s}{h}\right)}{\sqrt{v^2 + (h - fg)^2}} \right), \\ -\frac{1}{g}\cos\theta\sin\left(\frac{s}{h}\right) + \sin\theta \frac{(h - fg)\cos\left(\frac{s}{h}\right) + v\sin\left(\frac{s}{h}\right)}{\sqrt{v^2 + (h - fg)^2}} \end{array} \right),$$

$$\bar{\mathbf{U}} = -\sin\theta\left(1, \frac{1}{g}\cos\left(\frac{s}{h}\right), -\frac{1}{g}\sin\left(\frac{s}{h}\right)\right)$$

$$+ \cos\theta\left(0, \frac{(h - fg)\sin\left(\frac{s}{h}\right) - v\cos\left(\frac{s}{h}\right)}{\sqrt{v^2 + (h - fg)^2}}, \frac{(h - fg)\cosh\left(\frac{s}{h}\right) + v\sin\left(\frac{s}{h}\right)}{\sqrt{v^2 + (h - fg)^2}}\right),$$

$$\mathbf{V} = \bar{\mathbf{V}} = \frac{1}{\sqrt{v^2 + (h - fg)^2}} \left(\begin{array}{c} 0, \left((h - fg)\cos\left(\frac{s}{h}\right) + v\sin\left(\frac{s}{h}\right)\right), \\ -\left((h - fg)\sin\left(\frac{s}{h}\right) - v\cos\left(\frac{s}{h}\right)\right) \end{array} \right).$$

It is easily seen that the product of relative torsions $(\tau_g)_\alpha$ and $(\bar{\tau}_g)_\alpha$ is constant $\left((\tau_g)_\alpha(\bar{\tau}_g)_\alpha = \frac{1}{h^2 - \lambda^2}\right)$.

Example 5.2. Let us now consider the ruled surface of type I in pseudo-Galilean space G_1^3. It can be parameterized by

$$\Phi(s, v) = \left(s, f\sinh\left(\frac{s}{h}\right), f\cosh\left(\frac{s}{h}\right)\right) + v\left(1, \frac{1}{g}\cosh\left(\frac{s}{h}\right), \frac{1}{g}\sinh\left(\frac{s}{h}\right)\right),$$

where $f, g, h \in \mathcal{R}$, $f \neq 0$, $g \neq 0$, $h \neq 0$ and $|fg - h| > |v|$. Its striction curve $\beta(s) = \left(s,\ f \sinh\left(\frac{s}{h}\right),\ f \cosh\left(\frac{s}{h}\right)\right)$ does not lie in a pseudo-Galilean plane and $A(s) = \left(1, \frac{1}{g}\cosh\left(\frac{s}{h}\right), \frac{1}{g}\sinh\left(\frac{s}{h}\right)\right)$ is a unit generator non-isotropic vector field. From equations (4.1), (4.2) and (4.3), the Frenet apparatus of $\beta(s)$ is given by

$$A(s) = \mathbf{t} = \left(1, \frac{1}{g}\cosh\left(\frac{s}{h}\right), \frac{1}{g}\sinh\left(\frac{s}{h}\right)\right),$$

$$\mathbf{n} = \left(0, \sinh\left(\frac{s}{h}\right), \cosh\left(\frac{s}{h}\right)\right),$$

$$\mathbf{b} = \left(0, \cosh\left(\frac{s}{h}\right), \sinh\left(\frac{s}{h}\right)\right),$$

$$\kappa_\beta = \frac{1}{gh}, \quad \tau_\beta = \frac{1}{h}.$$

From (3.5), the isotropic timelike normal vector of Φ is expressed as

$$\mathbf{U} = \frac{\left(0, (fg - h)\sinh\left(\frac{s}{h}\right) + v\cosh\left(\frac{s}{h}\right), (fg - h)\cosh\left(\frac{s}{h}\right) + v\sinh\left(\frac{s}{h}\right)\right)}{\sqrt{(fg - h)^2 - v^2}}.$$

If φ is the hyperbolic angle between **U** and **n**, then it is given by

$$\langle \mathbf{U}, \mathbf{N} \rangle = -|\mathbf{U}||\mathbf{N}|\cosh\varphi,$$

it follows that

$$\cosh\varphi = \frac{(fg - h)}{\sqrt{(fg - h)^2 - v^2}}, \quad \sinh\varphi = \frac{v}{\sqrt{(fg - h)^2 - v^2}}.$$

Furthermore, using equation (3.5), the vector **V** is

$$\mathbf{V} = \frac{1}{\sqrt{(fg - h)^2 - v^2}} \begin{pmatrix} 0, -\left((fg - h)\cosh\left(\frac{s}{h}\right) + v\sinh\left(\frac{s}{h}\right)\right), \\ -\left((fg - h)\sinh\left(\frac{s}{h}\right) + v\cosh\left(\frac{s}{h}\right)\right) \end{pmatrix}.$$

Here, assuming that $\overline{\beta}(\overline{s})$ is the Bertrand mate of β, then it can be written as follows:

$$\begin{aligned}\overline{\beta}(\overline{s}) &= \beta(s) + \lambda(s)\mathbf{V}\\ &= \left(s,\ f\sinh\left(\frac{s}{h}\right),\ f\cosh\left(\frac{s}{h}\right)\right)\\ &\quad + \frac{\lambda}{\sqrt{(fg-h)^2 - v^2}}\begin{pmatrix} 0, -\left((fg-h)\cosh\left(\frac{s}{h}\right) + v\sinh\left(\frac{s}{h}\right)\right), \\ -\left((fg-h)\sinh\left(\frac{s}{h}\right) + v\cosh\left(\frac{s}{h}\right)\right)\end{pmatrix}.\end{aligned}$$

From equation (4.7), we get

$$(\tau_g)_\beta = \frac{1}{h}.$$

Also, equations (4.9) and (4.10) give

$$\cosh\Theta = \frac{ds}{d\overline{s}} = \frac{h}{\sqrt{h^2-\lambda^2}}, \quad \sinh\Theta = \lambda(\tau_g)_\beta \frac{ds}{d\overline{s}} = \frac{\lambda}{\sqrt{h^2-\lambda^2}}.$$

Using Proposition 4.1 to obtain

$$(\kappa_g)_\beta = -\frac{\lambda}{h^2}.$$

Now, from equation (4.6), the curvatures $(\overline{\tau}_g)_\beta$, $(\overline{\kappa}_g)_\beta$ and $(\overline{\kappa}_n)_\beta$ of $\overline{\beta}(\overline{s})$ can be computed as follows:

$$(\overline{\tau}_g)_\beta = -\frac{h}{(h^2-\lambda^2)}, \quad (\overline{\kappa}_g)_\beta = -\frac{\lambda}{(h^2-\lambda^2)}, \quad (\overline{\kappa}_n)_\beta = (\kappa_n)_\beta = 0.$$

The Darboux frame $\{\overline{\mathbf{t}}, \overline{\mathbf{U}}, \overline{\mathbf{V}}\}$ is expressed as:

$$\overline{\mathbf{t}} = \begin{pmatrix} \cosh\Theta, \left(\frac{1}{g}\cosh\Theta\cosh\left(\frac{s}{h}\right) + \sinh\Theta\frac{(fg-h)\sinh\left(\frac{s}{h}\right) + v\cosh\left(\frac{s}{h}\right)}{\sqrt{(fg-h)^2-v^2}}\right), \\ \frac{1}{g}\cosh\Theta\sinh\left(\frac{s}{h}\right) + \sinh\Theta\frac{(fg-h)\cosh\left(\frac{s}{h}\right) + v\sinh\left(\frac{s}{h}\right)}{\sqrt{(fg-h)^2-v^2}}\end{pmatrix},$$

$$\overline{\mathbf{U}} = \sinh\Theta\left(1, \frac{1}{g}\cosh\left(\frac{s}{h}\right), \frac{1}{g}\sinh\left(\frac{s}{h}\right)\right)$$

$$+ \cosh\Theta\left(0, \frac{(fg-h)\sinh\left(\frac{s}{h}\right) + v\cosh\left(\frac{s}{h}\right)}{\sqrt{(fg-h)^2 - v^2}},\right.$$

$$\left.\frac{(fg-h)\cosh\left(\frac{s}{h}\right) + v\sinh\left(\frac{s}{h}\right)}{\sqrt{(fg-h)^2 - v^2}}\right),$$

$$\mathbf{V} = \overline{\mathbf{V}} = \frac{1}{\sqrt{(fg-h)^2 - v^2}}\begin{pmatrix} 0, -\left((fg-h)\cosh\left(\frac{s}{h}\right) + v\sinh\left(\frac{s}{h}\right)\right), \\ -\left((fg-h)\sinh\left(\frac{s}{h}\right) + v\cosh\left(\frac{s}{h}\right)\right) \end{pmatrix}.$$

From previous computations, it is easily seen that the product of relative torsions $(\tau_g)_\beta$ and $(\overline{\tau}_g)_\beta$ is constant $\left((\tau_g)_\beta(\overline{\tau}_g)_\beta = -\frac{1}{h^2 - \lambda^2}\right)$.

6. Conclusion

The paper defines and characterizes Bertrand curves in G^3 and G_1^3. Furthermore, the relations between the geodesic curvatures, the normal curvatures and the relative torsions of these curves are obtained. Finally, we support our results through examples.

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